Int J Theor Phys (2008) 47: 3130–3138
DOI 10.1007/s10773-008-9746-6

# Fidelity of Two-Particle Wave Packets Moving around the Schwarzschild Spacetime

**B. Nasr Esfahani · M. Ahmadi · F. Ahmadi**

Received: 11 February 2008 / Accepted: 14 April 2008 / Published online: 1 May 2008


**Abstract** Fidelity for two-particle wave packets of spin-$\frac{1}{2}$ particles moving around the Schwarzschild spacetime is discussed. Both acceleration and gravity cause to produce a Wigner rotation that transforms the wave packet as it moves along a specified path in the gravitational field. For considered circular paths, the fidelity between the spin parts of initial and final states of the system, called the spin fidelity, is obtained as a function of angular velocity, elapsed proper time and radius of circular paths. For fixed elapsed proper time and angular momentum of the centroid, there always exists one circular orbit with determined radius on which the fidelity of spin parts is minimum. Using a numerical approach, the behavior of the spin fidelity in terms of the angular velocity, as well as the radius of paths is described for both the spin singlet and spin triplet states.

**Keywords** Local inertial frame · Wigner rotation · Two-particle wave packet · Fidelity · Schwarzschild spacetime

## 1 Introduction

Quantum information theory, that is, transmission, storage and processing of information using quantum mechanical systems, is now well developed. Many novel features of this theory relay on the entanglement and the non-locality associated with it. An early thought-provoking analysis of quantum composite system states was made by Einstein, Podolsky and Rosen who provided a specific argument for the incompleteness (not the incorrectness) of the quantum mechanical description of the microscopic world. They explicitly pointed out the surprising nature of entangled quantum systems that also introduced considerations of locality and realism in regard to microscopic physical systems [1, 2].

It is important to study all those processes that might have an effect on quantum information. For example, the spin of a massive particle, say an electron, is changed by Lorentz

B Nasr Esfahani (✉) · M. Ahmadi · F. Ahmadi
Department of Physics, Faculty of Sciences, University of Isfahan, Isfahan 81744, Iran
e-mail: ba_nasre@sci.ui.ac.ir

transformations and if the spin is used to handle a qubit of quantum information, that information will be modified by Lorentz transformations. Recently, a number of papers [3–6] have discussed how entanglement is affected by the Lorentz transformation in the jargon of special relativity. Relativistic quantum information theory may become a necessary theory in the near future.

Some efforts have been made to discuss the effect of gravitational field on quantum information. Terashima and Ueda extended the special relativistic considerations to general relativity [7] and discussed a mechanism of spin rotation caused by spacetime curvature for spin-$\frac{1}{2}$ particles moving in a gravitational field. It is shown that this effect gives rise to a spin entropy production that is unique to general relativity [8]. This means that even if the state of the particle is pure at one spacetime point, it becomes mixed at another spacetime point. The gravitational spin entropy production for particles with arbitrary spin moving in a curved spacetime is also discussed [9]. Also, the fidelity for the states of spin-$\frac{1}{2}$ particles moving in a traversable wormhole spacetime is discussed [10].

In the present article we extend the considerations of the recent paper [10] to discuss fidelity between the initial and final states of a bipartite quantum system consisting of two spin-$\frac{1}{2}$ particles (qubits) that are moving around the Schwarzschild spacetime. For both theoretical and potentially practical reasons, it may be interesting to quantify the fidelity for such a system. By inspecting the fidelity in the present problem, one can more closely investigate how a central massive body, for instance a black hole, affects a quantum communication that takes place around it. We will consider the wave packets of the system moving on circular paths, and discuss the fidelity for the spin singlet and spin triplet states, separately.

## 2 Transformation of Two-Particle Wave Packets Moving in the Schwarzschild Spacetime

In a curved spacetime the curvature causes to break the global rotational symmetry. Therefore, the spin of a particle in general relativity can be defined only locally by switching to an inertial frame at each point and then invoking the rotational symmetry of the local inertial frame. Now, a particle with spin $\frac{1}{2}$ in the curved spacetime is defined as a particle whose one-particle states furnish the spin-$\frac{1}{2}$ representation of the local Lorentz group. The one-particle momentum eigenstate is described as $|p,\sigma\rangle$ where $p=(\sqrt{|\mathbf{p}|^2+m^2c^2},\mathbf{p})$ is the four momentum of the particle as measured in the local inertial frame and $\sigma=\pm\frac{1}{2}$ denotes the $z$-components of the spin. Then the wave packet for a system of two non-interacting particles, observed in a local frame located at an initial point $x^{(i)}$, has the form

$$|\psi^{(i)}\rangle=\sum_{\sigma_1\sigma_2}\int d^3p_1\int d^3p_2 C_{\sigma_1\sigma_2}(p_1,p_2)|p_1,\sigma_1;p_2,\sigma_2\rangle, \tag{1}$$

where amplitudes $C_{\sigma_1\sigma_2}(p_1,p_2)$ determine the admixture of the two-particle momentum eigenstates $|p_1,\sigma_1;p_2,\sigma_2\rangle$ in the wave packet. Normalizing $|\psi^{(i)}\rangle$ to unity implies

$$\sum_{\sigma_1}\sum_{\sigma_2}\int d^3p_1\int d^3p_2|C_{\sigma_1\sigma_2}(p_1,p_2)|^2=1, \tag{2}$$

provided that $\langle p_1',\sigma_1';p_2',\sigma_2'|p_1,\sigma_1;p_2,\sigma_2\rangle=\delta^3(\mathbf{p_1'}-\mathbf{p_1})\delta^3(\mathbf{p_2'}-\mathbf{p_2})\delta_{\sigma_1'\sigma_1}\delta_{\sigma_2'\sigma_2}$.

Naturally, the wave packet (1) has two corresponding centroids that can describe a classical motion for the particles. Consider the motion of one of the centroids along a specified path $x^\mu(\tau)$ in the curved spacetime. Let $u^\mu(x)$ be the four velocity of the centriod as measured in general frame and $q^a(x)$ and $a^a(x)$ be the four momentum and the four accelerator of that centroid measured in the local inertial frame, respectively. The connection between general frame (Greek indices) and local frame (Latin indices) is furnished by the tetrad $e^a{}_\mu(x)$. It is required to assume that the spacetime curvature does not change drastically within the spacetime scale of the wave packet. Because of this assumption, we will consider motions that restrict distances between the centroids. In this situation we can use approximately one local inertial frame to discuss the motions of the two-particle system.

Accordingly, in the local inertial frame located at a final point $x^{(f)}$ the final wave packet will be

$$|\psi^{(f)}\rangle = \sum_{\sigma_1\sigma_2\sigma_1'\sigma_2'} \int d^3p_1 \sqrt{\frac{(\Lambda_1 p_1)^0}{p_1^0}} \int d^3p_2 \sqrt{\frac{(\Lambda_2 p_2)^0}{p_2^0}} C_{\sigma_1\sigma_2}(p_1, p_2) \\ \times D_{\sigma_1'\sigma_1}(W(\Lambda_1, p_1)) D_{\sigma_2'\sigma_2}(W(\Lambda_2, p_2)) |\Lambda_1 p_1, \sigma_1'; \Lambda_2 p_2, \sigma_2'\rangle, \tag{3}$$

where $\Lambda_1$ and $\Lambda_2$ are associated Lorentz transformations, and $W(\Lambda_1, p_1)$ and $W(\Lambda_2, p_2)$ are corresponding Wigner rotations. Also, $D_{\sigma'\sigma}(W)$ indicates the two-dimensional representation of the Wigner rotation operators [11].

In the following, as the background curved spacetime, we consider the Schwarzschild spacetime

$$ds^2 = -c^2\left(1 - \frac{r_s}{r}\right)dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{4}$$

where $r_s = \frac{2GM}{c^2}$ is the Schwarzschild radius. We require to introduce a local inertial frame at each point in the spacetime. Therefore, we choose the tetrad

$$e_0{}^t = \frac{1}{c\sqrt{1 - \frac{r_s}{r}}}, \quad e_1{}^r = \sqrt{1 - \frac{r_s}{r}}, \quad e_2{}^\theta = \frac{1}{r}, \quad e_3{}^\phi = \frac{1}{r\sin\theta}, \tag{5}$$

with all the other components being zero.

We want to introduce the Wigner rotation operator $W(\Lambda(x_f, x_i), p)$ for wave packets of spin-$\frac{1}{2}$ particles moving in the spacetime (4). Conveniently, we confine the argument to circular motions of wave packet around the center. In order to confine the extension of the wave packet, we suppose that both the centroids are moving with the same constant speed $v$ on a circle with radius $r$ around the black hole. Regarding the spherical symmetry of the metric, we can choose the plane of motion to be the equatorial plane $\theta = \frac{\pi}{2}$. Then, the components of the four-momentum of the centroids in the local inertial frame at any point are

$$q^0 = \gamma mc, \quad q^1 = q^2 = 0, \quad q^3 = \gamma mv. \tag{6}$$

Now using the results of the reference [9] for circular motions in a general spherically symmetric spacetime, we find the Wigner rotation operator for the present case as

$$W(\Lambda(\tau)) = e^{-iJ_2\Theta}, \tag{7}$$

where $J_2$ is the component of angular momentum operator along the 2-axes and

$$\Theta = 2\pi \left(\frac{\tau}{\tau_s}\right)\left(\frac{2z-3}{2z\sqrt{z}\sqrt{z-1}}\right) \times q\sqrt{q^2+1}\left(\sqrt{q^2+1} - \frac{qp}{\sqrt{p^2+1}+1}\right), \tag{8}$$

where $\tau = \tau_f - \tau_i$ is the finite elapsed proper time and $\tau_s$ is the proper time during which a photon rotates once on the circle of radius $r_s$. Also, $p = \frac{|\mathbf{p}|}{mc}$ which relates to the spatial momentum eigenvalue, $q = \frac{q^3}{mc} = \gamma \frac{v}{c}$ which relates to the angular velocity of each centroid, and $z = \frac{r}{r_s}$ measures the distance from the center of black hole. Evidently, the Wigner rotation operator (7) has a 2-dimensional representation as

$$D(\Theta) = \begin{pmatrix} \cos\frac{\Theta}{2} & \sin\frac{\Theta}{2} \\ -\sin\frac{\Theta}{2} & \cos\frac{\Theta}{2} \end{pmatrix}. \tag{9}$$

## 3 Spin Fidelity between the Initial and Final States

Suppose that in the Schwarzschild spacetime of previous section $|\psi^{(i)}\rangle$ denotes the state in a local inertial frame located at $x^i$ and $|\psi^{(f)}\rangle$ is the state in the local frame at final point $x^f$. Here it is convenient to discuss the fidelity $F$ between these initial and final states. The concept of fidelity is a basic ingredient in communication theory. For any given communication scheme the fidelity is a quantitative measure of the accuracy of the transmission. Fidelity ranges over the interval [0, 1] such that $F = 1$ indicates a perfect transmission. So we refer to $F = 1$ as perfect fidelity. Therefore, as $F \to 0$ the accuracy in transmission diminishes. In the following we confine the discussion to the fidelity between the spin parts of the states, which we call it the spin fidelity and indicate it by $F_s$. This can be achieved by using the corresponding initial and final reduced density matrices. To be more precise, in the following we choose two specific forms for the coefficient $C_{\sigma_1\sigma_2}(p_1, p_2)$ and proceed in two separate cases.

### 3.1 Spin Singlet State

In this case we naturally choose

$$C_{\sigma_1\sigma_2}(p_1, p_2) = \frac{1}{\sqrt{2}}\left(\delta_{\sigma_1\uparrow}\delta_{\sigma_2\downarrow} - \delta_{\sigma_1\downarrow}\delta_{\sigma_2\uparrow}\right) f(p_1) f(p_2), \tag{10}$$

where $f(p_1)$ and $f(p_2)$ are two normalized functions. Then the initial wave function (1) becomes

$$|\psi^{(i)}\rangle = \frac{1}{\sqrt{2}}\int d^3p_1 \int d^3p_2 f(p_1) f(p_2)[|p_1, \uparrow; p_2, \downarrow\rangle - |p_1, \downarrow; p_2, \uparrow\rangle]. \tag{11}$$

The related density matrix is $\rho^{(i)} = |\psi^{(i)}\rangle\langle\psi^{(i)}|$, which as we take its trace over the momentum, we obtain the following reduced density matrix

$$\varrho^{(i)} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{12}$$

which is a pure density matrix. Substituting (10) in (3) and taking the trace of $\rho^{(f)} = |\psi^{(f)}\rangle\langle\psi^{(f)}|$ over the momentum, we obtain the final reduced density matrix as

$$\begin{aligned}\varrho^{(f)}_{\sigma_1'\sigma_2',\sigma_1\sigma_2} = \frac{1}{2}\int d^3p_1|f(p_1)|^2\int d^3p_2|f(p_2)|^2 \\ \times [D_{\sigma_1'\uparrow}(\Theta_1)D_{\sigma_2'\downarrow}(\Theta_2)D_{\sigma_1\uparrow}(\Theta_1)D_{\sigma_2\downarrow}(\Theta_2) \\ - D_{\sigma_1'\downarrow}(\Theta_1)D_{\sigma_2'\uparrow}(\Theta_2)D_{\sigma_1\uparrow}(\Theta_1)D_{\sigma_2\downarrow}(\Theta_2) \\ - D_{\sigma_1'\uparrow}(\Theta_1)D_{\sigma_2'\downarrow}(\Theta_2)D_{\sigma_1\downarrow}(\Theta_1)D_{\sigma_2\uparrow}(\Theta_2) \\ + D_{\sigma_1'\downarrow}(\Theta_1)D_{\sigma_2'\uparrow}(\Theta_2)D_{\sigma_1\downarrow}(\Theta_1)D_{\sigma_2\uparrow}(\Theta_2)],\end{aligned} \tag{13}$$

where $D_{\sigma_1\sigma_2}(\Theta_1)$ or $D_{\sigma_1\sigma_2}(\Theta_2)$ is given by (9). Moreover, $\Theta_1$ or $\Theta_2$ is given by (8) with $p$ and $q$ corresponding to each particle.

To calculate the spin fidelity we use the equation [12]

$$F_s = \left[\mathrm{Tr}\left(\sqrt{\varrho^{(i)}\varrho^{(f)}}\right)\right]^2 = \left(\sqrt{\nu_1}+\sqrt{\nu_2}+\sqrt{\nu_3}+\sqrt{\nu_1}\right)^2, \tag{14}$$

where $\nu_1, \nu_2, \nu_3$ and $\nu_4$ are the eigenvalues of $\sqrt{\varrho^{(i)}\varrho^{(f)}}$. After doing some manipulations, we obtain

$$F_s = \frac{1}{4}\left(\varrho^{(f)}_{\uparrow\downarrow,\uparrow\downarrow} - \varrho^{(f)}_{\downarrow\uparrow,\uparrow\downarrow} - \varrho^{(f)}_{\uparrow\downarrow,\downarrow\uparrow} + \varrho^{(f)}_{\downarrow\uparrow,\downarrow\uparrow}\right). \tag{15}$$

This gives us

$$F_s = \frac{1}{2}\left(1 + \overline{\cos\Theta}^2 + \overline{\sin\Theta}^2\right), \tag{16}$$

where the overline is defined as $\overline{X} = \int d^3p|f(p^a)|^2X(p^a)$ which denotes the average over the momentum distribution. In order to evaluate the averages we choose a Gaussian form for $f(p)$, that is

$$f(p) = \frac{\sqrt{\alpha\delta(p^1)\delta(p^2)}}{\sqrt{\sqrt{\pi}mc}}\exp\left[-\frac{\alpha^2(p^3-q^3)^2}{2m^2c^2}\right]. \tag{17}$$

By this choice we reach to

$$\overline{\cos\Theta} = \frac{\alpha}{\sqrt{\pi}}\int dp e^{-\alpha^2(p-q)^2}\cos\Theta, \tag{18}$$

with a similar expression for $\overline{\sin\Theta}$, where $p = \frac{p^3}{mc}$. Obviously, $\overline{\cos\Theta}$ and $\overline{\sin\Theta}$ vanish as $q \to \infty$, resulting $F_s = \frac{1}{2}$. Substituting $\Theta$ from (8), we see that there is no analytical solution for the integral (18). However, some important results can be attained from a numerical approach.

In Fig. 1, the spin fidelity (16) is numerically sketched versus a varying $q$ but fixed $\tau$ and $z$. Recall that $q$ relates to angular velocity of the wave packet on the orbit. We see that $F_s$ descends uniformly from 1 to an asymptotic value $\frac{1}{2}$. That in this case the value of spin fidelity never becomes less than $\frac{1}{2}$ is a remarkable result. More interesting here is to sketch the behavior of the spin fidelity (16) in terms of radius of the orbits. Figure 2 shows such a sketch, that is $F_s$ versus $z$ while $q$ and $\tau$ are fixed. Just at $z = \frac{3}{2}$ the spin fidelity equals 1, suggesting a perfect communication. There exists one circle with $z \simeq 2.682$ on which $F$

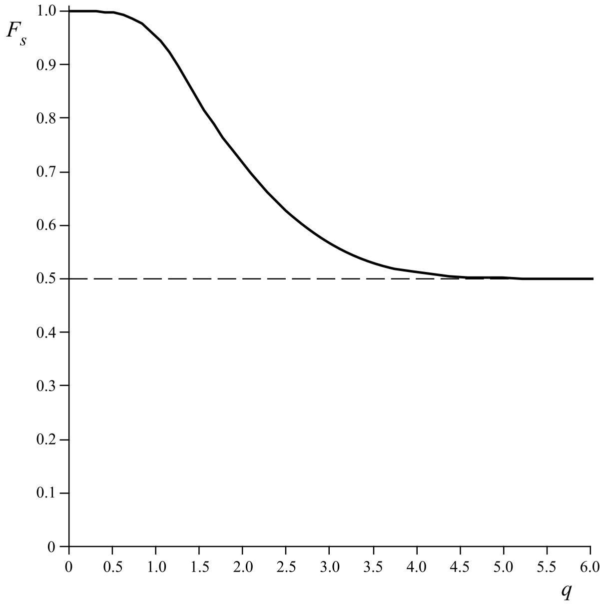



**Fig. 1** A sketch of $F_s$ given by (16) for a fixed $z$ and $\tau$ but varying $q$, in the spin singlet case. The curve descends uniformly from 1 and reaches to $\frac{1}{2}$, asymptotically. It never becomes less than 0.5

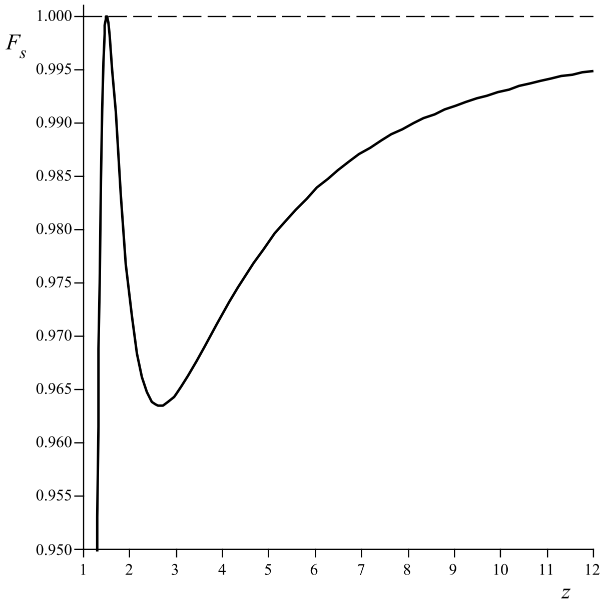



**Fig. 2** A sketch of $F_s$ for fixed $q$ and $\tau$ but varying $z$, in the spin singlet case. Just at $z = \frac{3}{2}$ the spin fidelity equals 1, suggesting a perfect communication. There exists one circle with $z \simeq 2.682$ on which $F_s$ is minimum, while it uniformly reaches to 0.5 at the horizon with $z = 1$

takes a minimum value of $F_s \simeq 0.963$ which is not very significant. As a consequence of asymptotic flatness of the metric as $z \to \infty$, $F_s \to 1$.

It must be noted that just on the event horizon (if is accessible) where $z = 1$ the Wigner rotation angle $\Theta$ diverges and so we encounter to an essential difficulty. However, it can be proved that the spin fidelity (16) approaches a finite value 0.5 near the event horizon, though

Fig. 2 does not show this. Evidently, inside the event horizon where $z < 1$, our argument is invalid.

### 3.2 Spin Triplet State

As another interesting choice and in order to compare the results, let

$$C_{\sigma_1\sigma_2}(p_1, p_2) = \frac{1}{\sqrt{2}}\left(\delta_{\sigma_1\uparrow}\delta_{\sigma_2\downarrow} + \delta_{\sigma_1\downarrow}\delta_{\sigma_2\uparrow}\right) f(p_1) f(p_2), \tag{19}$$

which as substituted in (1), leads to the following initial reduced density matrix

$$\varrho^{(i)} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{20}$$

Then, we have

$$\begin{aligned} \varrho^{(f)}_{\sigma_1'\sigma_2',\sigma_1\sigma_2} = {} & \frac{1}{2}\int d^3p_1|f(p_1)|^2 \int d^3p_2|f(p_2)|^2 \\ & \times [D_{\sigma_1'\uparrow}(\Theta_1)D_{\sigma_2'\downarrow}(\Theta_2)D_{\sigma_1\uparrow}(\Theta_1)D_{\sigma_2\downarrow}(\Theta_2) \\ & + D_{\sigma_1'\downarrow}(\Theta_1)D_{\sigma_2'\uparrow}(\Theta_2)D_{\sigma_1\uparrow}(1)D_{\sigma_2\downarrow}(\Theta_2) \\ & + D_{\sigma_1'\uparrow}(\Theta_1)D_{\sigma_2'\downarrow}(\Theta_2)D_{\sigma_1\downarrow}(\Theta_1)D_{\sigma_2\uparrow}(\Theta_2) \\ & + D_{\sigma_1'\downarrow}(\Theta_1)D_{\sigma_2'\uparrow}(\Theta_2)D_{\sigma_1\downarrow}(\Theta_1)D_{\sigma_2\uparrow}(\Theta_2)]. \end{aligned} \tag{21}$$

Substituting these density matrices in (14) and after doing some manipulations, we obtain the spin fidelity as

$$F_s = \frac{1}{2}\left(1 + \overline{\cos\Theta}^2 - \overline{\sin\Theta}^2\right), \tag{22}$$

which differs from (16) up to a minus sign.

To calculate the averages, we use the same $f(p)$ given by (17) and inevitably, again follow a numerical approach. Figure 3 shows the variation of $F_s$ given by (22) in terms of $q$. In comparison with the smooth curve of Fig. 1, this curve executes an aperiodic oscillation that asymptotically reaches to $\frac{1}{2}$. This contrast demonstrates significantly the difference between the behavior of spin fidelity for the spin triplet case and that of the spin singlet case. Moreover, Fig. 4 depicts the variation of (22) in terms of $z$, the distance from the center. We see that the least minimum occurs at $z \simeq 1.304$ where $F_s \simeq 0.054$ which is very significant. Again $F_s$ becomes the unity just at $\frac{3}{2}$. It must be noted that for the larger values of $z$ this curve returns up and asymptotically approaches to 1. On the other hand, as $z \to 1$, the curve oscillates more rapidly, meanwhile, damps to a finite value of 0.5.

In the spin singlet case discussed in Sect. 3.1 the spin fidelity never becomes less than 0.5 (see Figs. 1 and 2). While, in the spin triplet case the spin fidelity reaches to the values less than 0.5 (see Figs. 3 and 4).

## 4 Conclusion

We considered a quantum communication scheme around a central massive body (say a black hole), based on a system of two qubits moving on a circular path. It was desired to

**Fig. 3** A sketch of $F_s$ given by (22) for fixed $z$ and $\tau$ but varying $q$, in the spin triplet case. This demonstrates an aperiodic damping oscillation with an asymptotic value of 0.5

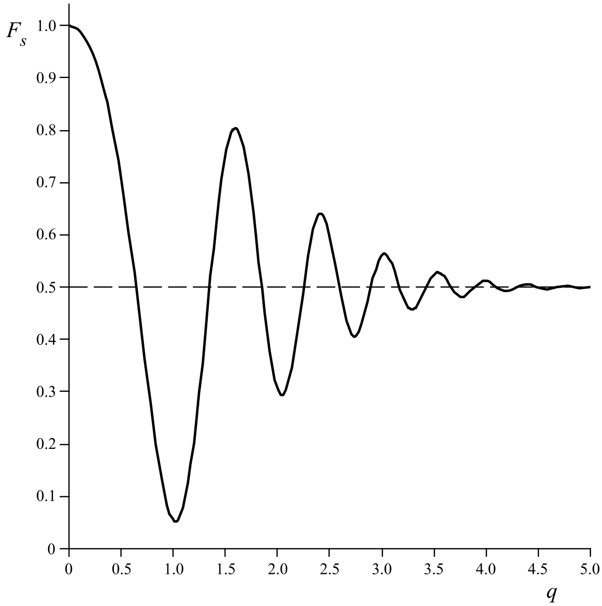
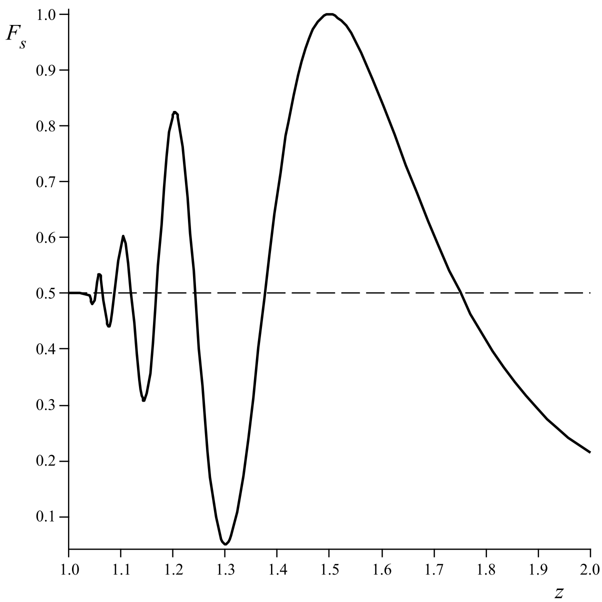



**Fig. 4** A sketch of $F_s$ for fixed $q$ and $\tau$ but varying $z$, in the spin triplet case. As $z \to 1$, that is, as the orbit closes to the event horizon, $F_s$ oscillates more rapidly, meanwhile damps to 0.5



quantify the fidelity between the spin parts of the system as a function of angular velocity of the system and radius of the path. Inspecting the figures of the last section which depict the numerical calculations, apart from the detailed results which are discussed previously, this general result can be conceived that since the spin fidelity in the spin singlet case never becomes less than 0.5, the communication based on two qubits in spin singlet preparation can be done more accurate than that of the spin triplet preparation. For black holes that the event horizon is accessible our argument become invalid on and inside it.

## References

1. Podolsky, B., Einstein, A., Rosen, N.: Phys. Rev. D **47**, 777 (1935)
2. Bohm, D.: Quantum Theory. Dover, New York (1989)
3. Peres, A., Scudo, P.F., Terno, D.R.: Phys. Rev. Lett. **88**, 230402 (2002)
4. Gingrich, R.M., Adami, C.: Phys. Rev. Lett. **89**, 270402 (2002)
5. Alsing, P.M., Milborn, G.J.: Quantum Inf. Comput. **2**, 487 (2002)
6. Li, H., Du, J.: Phys. Rev. A **68**, 022108 (2003)
7. Terashima, H., Ueda, M.: Phys. Rev. A **69**, 032113 (2004)
8. Terashima, H., Ueda, M.: J. Phys. A **38**, 2029 (2005)
9. Nasr, B., Dehdashti, S.: Int. J. Theor. Phys. **45**(6), 1495–1505 (2007)
10. Nasr Esfahani, B.: Int. J. Theor. Phys. **47**(5), 1373–1385 (2008)
11. Weinberg, S.: The Quantum Theory of Fields. Cambridge University Press, Cambridge (1995)
12. Hubner, M.: Phys. Lett. A **179**, 226–230 (1993)